Infinity in 45 Minutes or Less

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- About Nigerian money: "Meanwhile, the older coin denominations of 25kobo, 10kobo and 1kobo were quietly withdrawn from the system as their infinitely small purchasing power made them useless for transactions."
- About college football: "By joining the Pac 10, Utah's ability to get to a BCS Bowl got infinitely harder."

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In these cases, infinity means "really big" or "really small". That's not <u>really</u> what infinity means, however.

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Zeno's Paradox

About 100 years before Aristotle, Zeno posed the following paradox:

In order to cross a room, you have to first cross halfway across. Once you've done that, you must cross half of the remaining distance. Once you've done that, you must cross half of the remaining distance, and so on.

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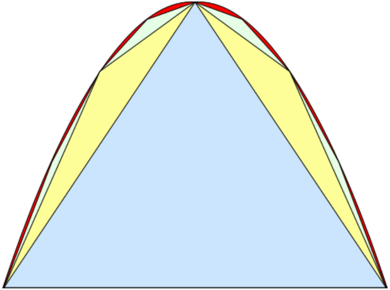
This may not be all that convincing. It would be nice if this was made more concrete.

Archimedes to the rescue!

The great mathematician Archimedes was the first to try to formalize this. He was trying to solve the following problem: how much area is there underneath an arc?



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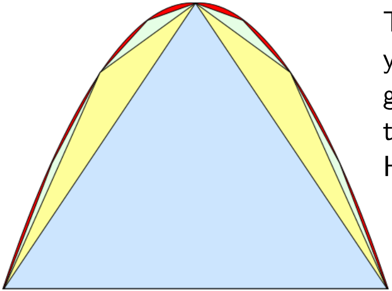

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Then Archimedes was able to show that the yellow triangles have a total area of $\frac{1}{4}$, the green triangles have a total area of $\frac{1}{16}$, the red triangles have a total area of $\frac{1}{64}$, and so on. He then reasoned that the area was given by:

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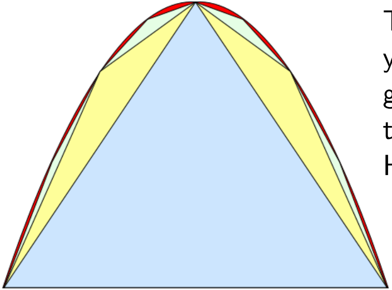

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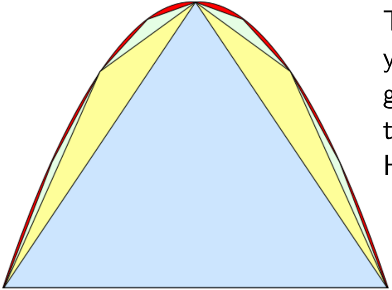

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$$\text{Area} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$$

Further, he was able to **prove** that this sum was equal to $\frac{4}{3}$.

Archimedes and infinite sums

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Over the years, many mathematicians have investigated these **infinite series**. A major point of interest is finding sufficient conditions for these infinite sums to be finite. The answers can be surprising: To give one famous example, proven by Oresme in the 1300s:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

This sum diverges, i.e. it is not finite.

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It may not seem like it, but this is an important result in cryptography (the RSA algorithm needs really big prime numbers).

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What was it? It must have been our original assumption that there are only finitely primes. So there are infinitely many primes.

□

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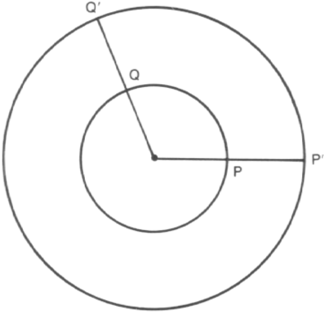

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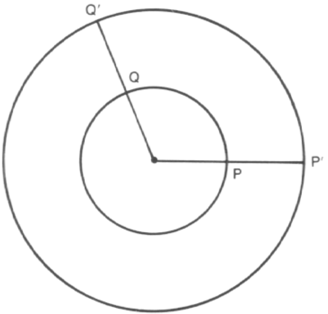

Galileo described the following paradox: take two concentric circles, and pick two different points on the outside circle, and draw the lines from those points to the center. You find that they intersect the smaller circle at different points. So even though the outside circle is bigger, **it's the same size as the the smaller circle**.

Galileo's Paradox

Another place Galileo encountered this was with square numbers:

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At this point, Galileo said that it was meaningless to talk about "infinite numbers", and that we couldn't say that one infinite number was bigger than another, or even that we couldn't say an infinite number is bigger than a finite number. Mathematicians continued to shun infinity for about 250 years.

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1. We often count objects by pairing them up with another set. We call this a **one-to-one correspondence** or a **bijection**.
2. An infinite set can be the same size as some part of it. This of course doesn't hold for finite sets. But, as Galileo showed (but didn't acknowledge), it **does** hold for infinite sets. In fact, this is one way of defining an infinite set.

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Pairing up infinite sets

The set of natural numbers has the same size as the set of square numbers.

$$
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The set of even numbers has the same size as the set of odd numbers.

$$\begin{array}{cccccc} 0 & 2 & 4 & 6 & 8 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 3 & 5 & 7 & 9 & \cdots \end{array}$$

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We say a set is **countable** if it can be paired up with the natural numbers (i.e. you can put the elements of the set in a list).

Even the rationals are countable!

The set of natural numbers has the same size as the set of rational numbers (i.e. fractions).

$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\dots$
$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\dots$
$\frac{0}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\dots$
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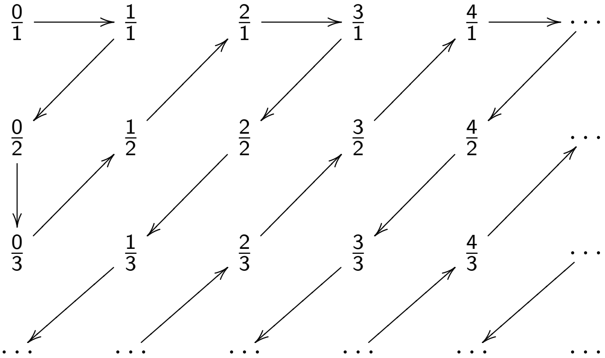
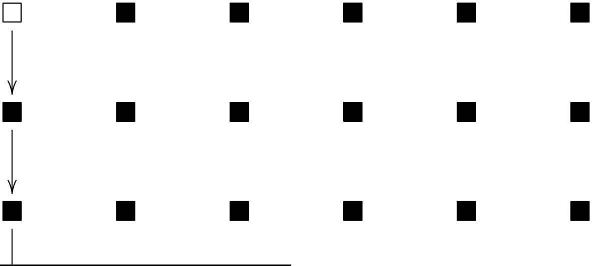


Hilbert's Classroom

Let's do the following thought experiment (this is based on Hilbert's Hotel): Imagine you walk into a classroom that has infinitely many rows of desks and you notice that every desk is filled. Now suppose that a student shows up late. If there were only finitely many seats, that student would be forced to stand. . .

Hilbert's Classroom

Let's do the following thought experiment (this is based on Hilbert's Hotel): Imagine you walk into a classroom that has infinitely many rows of desks and you notice that every desk is filled. Now suppose that a student shows up late. If there were only finitely many seats, that student would be forced to stand... but since there are infinitely many rows, I can simply tell all the students in the first column to move back one seat, and now everyone has a seat:



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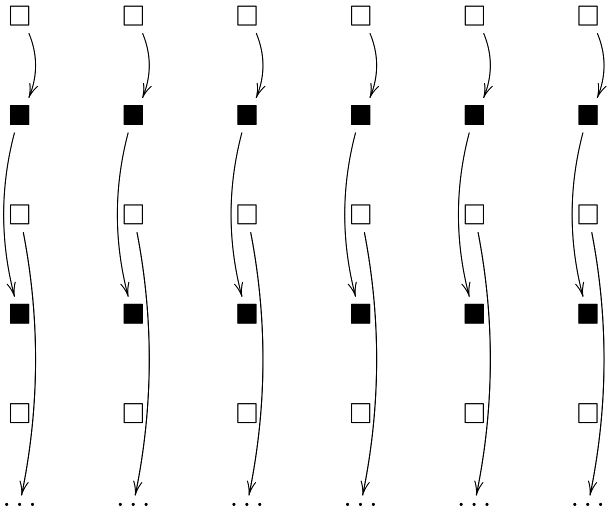

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Theorem (Cantor)

Let $\mathbb{R}$ be the set of all real numbers. Then $\mathbb{R}$ is not countable (i.e. the size of $\mathbb{R}$ is larger than the size of $\mathbb{N}$).

Proof.

Assume that $\mathbb{R}$ is countable. Let's make a list:

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Let $r = 0.1687\ldots$

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So there are different sizes of infinity. Cantor introduced some notation: he called the size of the natural numbers $\aleph_0$, and he called the size of the real numbers $\mathfrak{c}$. This proof says that $\mathfrak{c}$ is bigger than $\aleph_0$. (Incidentally, the symbol ∞ was created by John Wallis in 1655.)

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David Hilbert

"No one will drive us from the paradise which Cantor created for us."

Aftermath

Despite the initial pushback and controversy, mathematicians accepted Cantor's work over time. Partly this was due to mathematicians becoming more comfortable with Cantor's ideas, and partly because it helped answer some interesting questions (e.g. under what conditions is a function integrable?). It also inspired some brilliant mathematics which were used to describe such concepts as quantum mechanics and thermodynamics.

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It also led to a very interesting question: if the natural numbers are smaller than the real numbers ($\aleph_0 < \mathfrak{c}$), is there anything in between? In other words, can we find a set big enough that we can't pair it up with the natural numbers, but small enough so that we can't pair it up with the real numbers? Cantor guessed that there was nothing in between. This guess came to be called the **continuum hypothesis**.

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In other words, the assumptions we make about sets and the real numbers are not strong enough to determine whether it is true or not. A very vibrant area of research (still called set theory) is being done on what mathematical models can look like with or without the continuum hypothesis, the consequences of those models, placing them in a hierarchy, etc.

Further Reading

Wikipedia has good entries on infinity, one-to-one correspondence, Hilbert's Hotel, and Cantor.

Much of this can be found in 1,2,3, . . . , Infinity by George Gamow.

For a challenge, try Set Theory: An Introduction To Independence Proofs by Ken Kunen, which is a graduate level text describing the work of Gödel and Cohen.