

Research Statement

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1 Classical Model Theory

Classical model theory was started in the 1930s by Gödel in order to investigate the connection between systems of axioms, or theories, and models, or structures that satisfy those axioms. The field of classical model theory was greatly expanded by Tarski, Abraham Robinson, Vaught, Morley, Keisler, and many others. It has become a vibrant and growing area of study, with connections to many other areas of mathematics.

The model theory that I am investigating follows the notion of Chang and Keisler that “universal algebra + logic = model theory”. Universal algebra deals with algebraic structures along with homomorphisms, products, and substructures; model theory attempts to explain certain phenomena that arise by examining the axiomatizations of structures. For example, a class of models of a theory is closed under submodels if and only if it is axiomatizable by universal sentences. As another example, a class of models of a theory is closed under unions of chains if and only if it is axiomatizable by universal-existential sentences, that is, sentences of the form “for all x_1, x_2, \dots, x_m there exist y_1, y_2, \dots, y_n such that ...”. Thus, model theory is of great use in describing the properties of certain algebraic structures.

1.1 Model Companions

My thesis focused on a 1978 conjecture of Wheeler. This conjecture involves a correlation between model companions and universal Horn fragments. In order to describe these concepts, I use four common examples of theories: linear orders, partial orders, fields, and ordered fields. Each of these theories is contained in a special stronger theory, called a *model companion*. Respectively, the stronger theories are dense linear orders without endpoints, the theory of the random poset, algebraically closed fields, and real closed fields. The existence of a model companion for the theory of fields is essentially equivalent to the Nullstellensatz; the existence of a model companion for the theory of ordered fields is essentially equivalent to the Artin-Schreier theorem. Not all theories have a model companion.

What characterizes a model companion is that it is *model complete*. The idea of being model complete is related to the idea of being algebraically closed. A field \mathfrak{A} is algebraically closed if every polynomial with coefficients from \mathfrak{A} that has a root in an extension field \mathfrak{B} already has a root in \mathfrak{A} . Similarly, a theory is model complete if, for every model \mathfrak{A} of that theory, if an existential formula is witnessed in an extension model \mathfrak{B} of that theory, it is already witnessed in \mathfrak{A} .

Model complete theories are powerful in the sense that they are slight generalizations of theories that *admit quantifier elimination*. A theory admits quantifier elimination if, over the theory, each formula is equivalent to a formula that does not use any quantifiers. The four model companions above all happen to admit quantifier elimination. For example, in algebraically closed fields, the formula $\exists x(ax^2 + bx + c = 0)$ is equivalent to $(a = 0 \wedge b = 0 \wedge c = 0) \vee (a = 0 \wedge b \neq 0) \vee (a \neq 0)$.

1.2 Universal Horn Fragments

The second component of Wheeler's conjecture is *universal Horn fragments*. An axiom is universal Horn if it is equivalent to a sentence of the form:

$$\forall \mathbf{x}(\alpha_1(\mathbf{x}) \wedge \dots \wedge \alpha_n(\mathbf{x}) \rightarrow \alpha_0(\mathbf{x}))$$

where each $\alpha_i(\mathbf{x})$ is an atomic formula. A more intuitive characterization is that an axiom is universal Horn if it is preserved under submodels of products of models. In other words, an axiom φ is universal Horn if it is satisfied by submodels of products of models of φ . As an example, take \mathbb{Z} as the integers with $\leq^{\mathbb{Z}}$ the usual ordering, and let \mathfrak{A} be the model with domain $\mathbb{Z} \times \mathbb{Z}$, and let $\leq^{\mathfrak{A}}$ be the product order. The transitivity of $\leq^{\mathbb{Z}}$ is universal Horn: if $(a_1, a_2) \leq^{\mathfrak{A}} (b_1, b_2)$ and $(b_1, b_2) \leq^{\mathfrak{A}} (c_1, c_2)$, then $(a_1, a_2) \leq^{\mathfrak{A}} (c_1, c_2)$. An example of an axiom which is not universal Horn is the linearity of $\leq^{\mathbb{Z}}$: the elements $(0, 1)$ and $(1, 0)$ are not linearly ordered in \mathfrak{A} . The universal Horn fragment of a theory is the set of axioms of that theory which are universal Horn. For example, the universal Horn fragment of the theory of linear orders is the theory of partial orders.

1.3 Wheeler's Conjecture

Wheeler conjectured that, for a theory T , if T has a model companion, then the universal Horn fragment of T has a model companion[5]. For example, the theory of linear orders has as its model companion the theory of dense linear orders without endpoints, while the universal Horn fragment of linear orders has as its model companion the theory of the random poset. In 1980, Glass and Pierce showed this conjecture to be false, by showing it failed for the theory of linearly ordered abelian groups. [4]. However, the conjecture does hold true for many theories, for example, the theories of fields and ordered fields, over the appropriate languages. The proof that the universal Horn fragments of these and other theories have model companions, due to Cherlin, MacIntyre, Burris, Werner, and others, follows a construction technique first developed by Arens and Kaplansky [1]. This construction creates a certain Boolean sheaf whose stalks are models of the original theory T .

1.4 An Alternative to Wheeler's Conjecture

In my thesis, I worked with a theory of Boolean-indexed models over a larger language which simulates this Boolean sheaf, except that the underlying Boolean space is replaced by an atomless Boolean algebra. Coupled with this theory is the translation of a base theory T to an extension of the theory of Boolean-indexed models. In general, this theory does not capture all Boolean sheaf properties. However, it does give a model companion over this larger language if the original theory T is model complete. In my thesis, I also investigated the relationship between this new theory and the universal Horn fragment of the original theory. In this manner, I explored the connection between universal Horn fragments and model companions. I am currently preparing my thesis for publication, and my research is moving toward finding a necessary and sufficient condition for when the universal Horn fragment of a companionable theory has a model companion.

1.5 Heyting-Indexed Models

There is a natural direction in which the Boolean-indexed model construction may be generalized. Heyting algebras have a fundamental connection to the structure of intuitionistic logic. Just as the theory of Boolean algebras has a model companion, namely the theory of atomless Boolean algebras, the theory of Heyting algebras also has a model companion. Thus, a variation is to replace the underlying atomless Boolean algebras in Boolean-indexed models with the model companion of Heyting algebras. As a result, I may have a framework I can use to extend classical model theoretic results to models of intuitionistic logic.

2 Classical Model Theoretic Methods for Intuitionistic Logic

2.1 Intuitionistic Logic

Constructive mathematics has its origins in the philosophical considerations of intuitionists like Brouwer, and of constructivists like Markov, Bishop, and others. Just as classical logic is the formalization of traditional mathematical reasoning, intuitionistic logic is the formalization of constructive mathematical reasoning. The original formalization of intuitionistic logic is due to Heyting. Intuitionistic logic is often described as classical logic without the law of excluded middle.

So far, I only use intuitionistic logic as a formal generalization of classical logic, similar to group theory being a generalization of abelian group theory. Specifically, we use classical constructions of models of intuitionistic logic as generalizations of the usual classical models of classical logic. Examples are Kripke models and sheaf models, which have been investigated by many since the 1960s. Model theory of intuitionistic logic, while contributing many interesting results, has not been investigated to the same degree as classical model theory.

2.2 Completed Projects: Universal Sentences and Quantifier Elimination

I collaborate regularly with Jon Fleischmann, Dan McGinn, and Wim Ruitenburg. This has resulted in two published papers. In our first paper, we characterize an intuitionistic equivalent of classical universal sentences[2]. In our second paper, we construct a family of Kripke models whose intuitionistic theories admit quantifier elimination[3]. We currently collaborate on a project which examines universal-existential sentences in the context of intuitionistic logic. In particular, I am translating Kripke models into a first-order language, much like I translated Boolean sheaves into a classical language, and am investigating the types of sentences that correspond with universal-existential sentences in this translated language. In this way, we are actively extending concepts from classical model theory to models of intuitionistic logic.

3 Undergraduate Research

A goal of many liberal arts colleges is getting undergraduates involved in research. Of course, the most important part of a research project is a question that is at

the proper difficulty level for an undergraduate. My research has created several such questions. As mentioned above, I proved in my thesis that the translation of a model complete theory is model complete. One project would be to investigate the translation of theories that are known to be model complete (e.g. dense linear orders without endpoints). Doing research on these translations would allow students to conduct research in a burgeoning field of mathematics. A more involved project would be to look at a translation of the theory of linearly ordered abelian groups, which Glass and Pierce used to disprove Wheeler's Conjecture. It would be very interesting to examine why Wheeler's Conjecture failed to hold for that theory, but the translated theory is model complete. This question would likely need to be used for a summer research project, or for a two-semester long special study. In addition to these ideas, my membership in the MAA allows me to keep updated on interesting undergraduate research.

Beyond this, I have taken many steps to get prepared for working with students in conducting research. At the Joint Mathematical Meetings in 2009, I took an MAA short course in order to familiarize myself with many of the basic strategies, such as holding regular meetings and setting firm, specific goals. These ideas match my experience in an REU at Trinity University during the summer of 2000. Knowing what it is like for an undergraduate conducting research, I understand that professors need to be very involved with their students. An active advisor is necessary in order for the project can be a success, both in terms of the results achieved as well as helping the student understand the process of mathematical research.

4 Summary

My research promises to produce a number of results related to model completeness and universal Horn fragments in classical model theory. One goal is to use my work on Boolean-indexed models to obtain a characterization of when Wheeler's conjecture holds. Further, I plan to generalize these methods to extend classical model theory to models of intuitionistic logic. I have already started investigating these generalizations, and am sure that there are many promising results ahead. This is an area of mathematics which is relatively unknown. I am excited to continue research into this promising field.

References

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